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LOCAL DISCONTINUOUS GALERKIN METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS WITH HIGHER ORDER DERIVATIVES*

JUE YAN† AND CHI-WANG SHU‡

Abstract. In this paper we review the existing and develop new local discontinuous Galerkin methods for solving time dependent partial differential equations with higher order derivatives in one and multiple space dimensions. We review local discontinuous Galerkin methods for convection diffusion equations involving second derivatives and for KdV type equations involving third derivatives. We then develop new local discontinuous Galerkin methods for the time dependent bi-harmonic type equations involving fourth derivatives, and partial differential equations involving fifth derivatives. For these new methods we present correct interface numerical fluxes and prove L^2 stability for general nonlinear problems. Preliminary numerical examples are shown to illustrate these methods. Finally, we present new results on a post-processing technique, originally designed for methods with good negative-order error estimates, on the local discontinuous Galerkin methods applied to equations with higher derivatives. Numerical experiments show that this technique works as well for the new higher derivative cases, in effectively doubling the rate of convergence with negligible additional computational cost, for linear as well as some nonlinear problems, with a local uniform mesh.

Key words. discontinuous Galerkin method, partial differential equations with higher derivatives, stability, error estimate, post-processing

Subject classification. Applied and Numerical Mathematics

1. Introduction. In this paper we review existing and develop new local discontinuous Galerkin methods for solving time dependent partial differential equations with higher order derivatives in one and multiple space dimensions. We consider a sequence of such partial differential equations with increasingly higher order derivatives. A hyperbolic conservation law

$$U_t + \sum_{i=1}^{d} f_i(U)_{x_i} = 0 \tag{1.1}$$

is a partial differential equation with first derivatives. The convection diffusion equation

$$U_t + \sum_{i=1}^{d} f_i(U)_{x_i} - \sum_{i=1}^{d}\sum_{j=1}^{d} (a_{ij}(U)U_{x_j})_{x_i} = 0 \tag{1.2}$$

where $(a_{ij}(U))$ is a symmetric, semi-positive definite matrix, is a partial differential equation with second derivatives. The general KdV type equation

$$U_t + \sum_{i=1}^{d} f_i(U)_{x_i} + \sum_{i=1}^{d} \left(r_i'(U) \sum_{j=1}^{d} g_{ij}(r_i(U)_{x_i})_{x_j} \right)_{x_i} = 0 \tag{1.3}$$

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is a partial differential equation with third derivatives. The time dependent bi-harmonic equation

$$U_t + \sum_{i=1}^{d} f_i(U)_{x_i} + \sum_{i=1}^{d} (a_i(U_{x_i})U_{x_i x_i})_{x_i x_i} = 0 \tag{1.4}$$

is a partial differential equation with fourth derivatives, where the nonlinearity could be more general but we just present (1.4) as an example. The following equation

$$U_t + \sum_{i=1}^{d} f_i(U)_{x_i} + \sum_{i=1}^{d} g_i(U_{x_i x_i})_{x_i x_i x_i} = 0 \tag{1.5}$$

is a partial differential equation with fifth derivatives, where again the nonlinearity could be more general but we just present (1.5) as an example. Similar equations with sixth or higher derivatives could also be presented. All these equations, and their time independent steady state counterparts, appear often in physical and engineering applications. In this paper we use capital letters U etc. to denote the solutions to the PDEs and lower case letters to denote the numerical solutions.

The type of discontinuous Galerkin methods we will discuss in this paper, using a discontinuous Galerkin finite element approximation in the spatial variables coupled with explicit, nonlinearly stable high order Runge-Kutta time discretization [19], were first developed for the conservation laws (1.1) containing first derivatives by Cockburn et al. in a series of papers [8, 9, 6, 4, 10]. We will briefly review this method in section 2.1. For a detailed description of the method as well as its implementation and applications, we refer the readers to the lecture notes [3], the survey paper [5], other papers in that Springer volume, and the review paper [12].

For equations containing higher order spatial derivatives, discontinuous Galerkin methods cannot be directly applied. This is because the solution space, which consists of piecewise polynomials discontinuous at the element interfaces, is not regular enough to handle higher derivatives. This is a typical "non-conforming" case in finite elements. A naive and careless application of the discontinuous Galerkin method directly to the heat equation containing second derivatives could yield a method which behaves nicely in the computation but is inconsistent with the original equation and has O(1) errors to the exact solution [18, 12].

The idea of local discontinuous Galerkin methods for time dependent partial differential equations with higher derivatives is to rewrite the equation into a first order system, then apply the discontinuous Galerkin method on the system. A key ingredient for the success of such methods is the correct design of interface numerical fluxes. These fluxes must be designed to guarantee stability and local solvability of all the auxiliary variables introduced to approximate the derivatives of the solution. The local solvability of all the auxiliary variables is why the method is called a "local" discontinuous Galerkin method in [11].

The first local discontinuous Galerkin method was developed by Cockburn and Shu [11], for the convection diffusion equation (1.2) containing second derivatives. Their work was motivated by the successful numerical experiments of Bassi and Rebay [1] for the compressible Navier-Stokes equations. Later, Yan and Shu [20] developed a local discontinuous Galerkin method for the general KdV type equation (1.3) containing third derivatives. In both [11] and [20], suitable numerical fluxes at element interfaces were given, which led to provable nonlinear L^2 stability of the methods as well as error estimates for the linear cases. Numerical examples were shown to illustrate the stability and accuracy of these methods. These results will be briefly reviewed in sections 2.2 and 2.3, to motivate the new development in later sections.

In section 3 of this paper we develop new local discontinuous Galerkin methods for the time dependent bi-harmonic type equation (1.4) involving fourth derivatives, and in section 4 we do the same thing for the

partial differential equation (1.5) involving fifth derivatives. Similar methods can be designed for well posed partial differential equations involving even higher derivatives. As before, we give recipes for correct inter-element numerical fluxes which lead to provable nonlinear L^2 stability of the methods. The stability result is independent of the coefficients in front of the higher order derivatives, hence the methods are especially suitable for the so-called "convection dominated" problems, i.e. those with small coefficients to the higher derivative terms and hence are dominated by the first derivative convection terms. Also, these methods are extremely local and hence efficient for parallel implementations and easy for h-p adaptivity. We will provide preliminary numerical examples to verify the stability and accuracy of these methods.

A post-processing technique was introduced in [7], which is a local convolution if the mesh is locally uniform and has been proven to be able to recover the solution to an accuracy of Δx^{2k+1} rather than the usual Δx^{k+1} when piecewise polynomials of degree k are used, for linear conservation laws (1.1) (theoretically proven) and for linear convection diffusion equations (1.2) (numerically observed). The post-processing is even useful in enhancing accuracies for some nonlinear cases in numerical experiments. The key ingredient of this post-processing technique is a negative-order error estimate, thus if the problem is linear, the solution is sufficiently smooth, and the method has a good negative-order error estimate, then the post-processing will achieve the order of convergence of the negative-order norm (which is usually bigger than the order of convergence of the L^2-norm of the error in Galerkin methods). In section 5 of this paper, we apply this post-processing technique to our new local discontinuous Galerkin method in [20] and in this paper, for the linear KdV like equations (1.3), linear time dependent bi-harmonic equations (1.4), and the linear PDE (1.5) containing fifth derivatives. We observe the same accuracy enhancement capability for all these cases, where accuracy has been increased from Δx^{k+1} before post-processing to Δx^{2k+1} or higher (in many cases Δx^{2k+2}) after post-processing. This strongly suggests that the LDG methods we have designed for these equations with higher derivatives have an order of convergence of $2k+1$ or higher in the negative-order norm, when piecewise polynomials of degree k are used. The post-processing technique thus is fully taking advantage of this. To demonstrate that this accuracy enhancement can even be observed for some nonlinear problems, we also apply it to the local discontinuous Galerkin method in [20] for the nonlinear KdV equations. As the post-processing step is applied only at the end of the computation and is applied only locally, the additional computational cost is negligible.

Concluding remarks summarizing results in this paper and indicating future work are included in section 6.

2. Review of the discontinuous Galerkin methods for PDEs with first, second and third derivatives. In this section we review the essential points of the discontinuous Galerkin methods for the conservation laws (1.1) with first derivatives, the local discontinuous Galerkin methods for the convection diffusion equations (1.2) with second derivatives and the KdV type equations (1.3) with third derivatives. For simplicity of presentation, we will use one dimensional cases to present the methods. However, we will indicate which results are also valid for the general multi-dimensional cases.

2.1. Conservation Laws. The one dimensional version of the conservation laws (1.1) has the form

$$U_t + f(U)_x = 0. \tag{2.1}$$

First let's introduce some notations. The computational mesh is given by $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, for $j = 1, ..., N$, with the center of the cell denoted by $x_j = \frac{1}{2}\left(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}\right)$ and the size of each cell by $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$. We will denote $\Delta x = \max_j \Delta x_j$. If we multiply (2.1) by an arbitrary test function $V(x)$,

integrate over the interval I_j, and integrate by parts, we get

$$\int_{I_j} U_t V dx - \int_{I_j} f(U) V_x dx + f(U(x_{j+\frac{1}{2}}, t)) V(x_{j+\frac{1}{2}}) - f(U(x_{j-\frac{1}{2}}, t)) V(x_{j-\frac{1}{2}}) = 0. \tag{2.2}$$

This is the starting point for designing the discontinuous Galerkin method. The semi-discrete version of the discontinuous Galerkin method [9, 6, 4, 10] can be described as follows: we replace both the solution U and the test function V by piecewise polynomials of degree at most k, and denote them by u and v. That is, $u, v \in \mathcal{V}_{\Delta x}$ where

$$\mathcal{V}_{\Delta x} = \{v : \ v \text{ is a polynomial of degree at most } k \text{ for } x \in I_j, \ \ j = 1, ..., N\}. \tag{2.3}$$

With this choice, there is an ambiguity in (2.2) in the last two terms involving the boundary values at $x_{j\pm\frac{1}{2}}$, as both the solution u and the test function v are *discontinuous* exactly at these boundary points. A crucial ingredient for the success of discontinuous Galerkin method for the conservation laws is the correct choice of the numerical fluxes to overcome (or we could say it in a positive way, to utilize) this ambiguity of discontinuity at the element interfaces. The idea is to treat these terms by an upwinding mechanism (information from characteristics), borrowed from the successful high resolution finite volume schemes. Thus $f(u)$ at the interface $x_{j+\frac{1}{2}}$ for each j is given by a single valued monotone numerical flux

$$\hat{f}_{j+\frac{1}{2}} = \hat{f}(u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}}), \tag{2.4}$$

which depends both on the left limit $u^-_{j+\frac{1}{2}}$ and on the right limit $u^+_{j+\frac{1}{2}}$ of the discontinuous numerical solution at the interface $x_{j+\frac{1}{2}}$. Here monotone flux means that the function $\hat{f}$ is a non-decreasing function of its first argument and a non-increasing function of its second argument. It is also assumed to be at least Lipschitz continuous with respect to each argument and to be consistent with the physical flux $f(u)$ in the sense that $\hat{f}(u, u) = f(u)$. If (2.1) is a system rather than a scalar equation, then the monotone flux is replaced by an exact or approximate Riemann solver, see, e.g. [16] for details. On the other hand, the test function v at the interfaces $x_{j\pm\frac{1}{2}}$ is taken from inside the cell I_j, namely $v^-_{j+\frac{1}{2}}$ and $v^+_{j-\frac{1}{2}}$ respectively. The final semi-discrete scheme

$$\int_{I_j} u_t v dx - \int_{I_j} f(u) v_x dx + \hat{f}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0 \tag{2.5}$$

is then discretized by a nonlinearly stable high order Runge-Kutta time discretizations [19]. Nonlinear TVB limiters [17] may be used if the solution contains strong discontinuities. The schemes thus obtained have the following attractive properties, for the general multi-dimensional case (1.1) with arbitrary triangulations:

1. It can be easily designed for any order of accuracy. In fact, the order of accuracy can be locally determined in each cell, thus allowing for efficient p adaptivity.
2. It can be used on arbitrary triangulations, even those with hanging nodes, thus allowing for efficient h adaptivity.
3. It is extremely local in data communications. The evolution of the solution in each cell needs to communicate only with the immediate neighbors, regardless of the order of accuracy, thus allowing for efficient parallel implementations. See, e.g. [2].

These schemes also have the following provable theoretical properties, all of these are valid also for the multi-dimensional case (1.1) with arbitrary triangulations:

1. The semi-discrete scheme (2.5), and certain time discretization of it, such as implicit backward Euler and Crank-Nicholson, have excellent nonlinear stability properties. One can prove a strong L^2

stability and a cell entropy inequality for the square entropy, for the general nonlinear scalar case (1.1), for any orders of accuracy on arbitrary triangulations in any space dimension, without the need for nonlinear limiters [14]. Notice that these stability results are valid even when the solution contains discontinuities such as shocks.

2. For linear problems with smooth solutions, these methods using piecewise polynomials of degree k have a provable error estimate of order $\Delta x^{k+\frac{1}{2}}$ in L^2 [15]. In effect, for most triangulations one could observe (and prove in many cases) convergence of the order Δx^{k+1} in both L^2 and L^∞ norms, for both linear and nonlinear problems.
3. When nonlinear TVB limiters [17, 9, 4] are used, the methods can be proven stable in the total variation norm for scalar one dimensional nonlinear problems (2.1), and stable in the L^∞ norm for scalar multi-dimensional nonlinear problems (1.1).

2.2. Convection diffusion equations. The one dimensional version of the convection diffusion equation (1.2) has the form

$$U_t + f(U)_x - (a(U)U_x)_x = 0, \tag{2.6}$$

where $a(U) \geq 0$.

The semi-discrete version of the local discontinuous Galerkin method for solving (2.6) [11] approximates the following lower order system

$$U_t + f(U)_x - (b(U)Q)_x = 0, \qquad Q - B(U)_x = 0, \tag{2.7}$$

where $b(U) = \sqrt{a(U)}$, and $B(U) = \int^U b(U)dU$. We can then *formally* use the same discontinuous Galerkin method for the convection equation to solve (2.7), resulting in the following scheme: find $u, q \in \mathcal{V}_{\Delta x}$ such that, for all test functions $v, w \in \mathcal{V}_{\Delta x}$,

$$\begin{aligned}
&\int_{I_j} u_t v dx - \int_{I_j} (f(u) - b(u)q)v_x dx + (\hat{f}_{j+\frac{1}{2}} - \hat{b}_{j+\frac{1}{2}}\hat{q}_{j+\frac{1}{2}})v^-_{j+\frac{1}{2}} - (\hat{f}_{j-\frac{1}{2}} - \hat{b}_{j-\frac{1}{2}}\hat{q}_{j-\frac{1}{2}})v^+_{j-\frac{1}{2}} = 0 \\
&\int_{I_j} qw dx + \int_{I_j} B(u)w_x dx - \hat{B}_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} + \hat{B}_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} = 0.
\end{aligned} \tag{2.8}$$

Again, a crucial ingredient for the method to be stable is the correct choice of the numerical fluxes (the "hats"). However, there is no longer a upwinding mechanism or characteristics to guide the design of these fluxes. In [11], criteria are given for these fluxes to guarantee stability and convergence. The best choice is to use the "alternating principle" in designing the fluxes:

$$\hat{b} = \frac{B(u^+) - B(u^-)}{u^+ - u^-}; \qquad \hat{q} = q^-; \qquad \hat{B} = B(u^+) \tag{2.9}$$

and $\hat{f}$ is chosen as before in (2.4). Notice that we did not write the subscript $j+\frac{1}{2}$ for the fluxes as they are all evaluated at this interface point. The "alternating principle" refers to the alternating choices of $\hat{q}$ and $\hat{B}$: if the left value is chosen for the former then the right value is chosen for the latter, as in (2.9). One could also choose

$$\hat{q} = q^+; \qquad \hat{B} = B(u^-)$$

with all the other fluxes unchanged. These choices of fluxes guarantee stability and optimal convergence.

We remark that the appearance of the auxiliary variable q is superficial: when a local basis is chosen in cell I_j then q is eliminated by using the second equation in (2.8) and solving a small linear system if the

local basis is not orthogonal. The actual scheme for u takes a form similar to that for convection alone. This is a big advantage of the scheme over the traditional "mixed methods", whose auxiliary variable is usually genuinely global. This is the reason that the scheme is termed *local* discontinuous Galerkin method in [11]. We also remark that the choice of fluxes in (2.9) by the alternating principle yields the most compact stencil for the scheme for u after the auxiliary variable q is locally eliminated.

The schemes thus designed for the one dimensional equation (2.6), or in fact for the most general multi dimensional nonlinear convection diffusion equations (1.2), which is nonlinear both in the first derivative convection part and in the second derivation diffusion part, retain *all* of the three attractive properties listed above for the method used on convection equations. They also have the following provable theoretical properties, all of these are valid for the multi-dimensional case (1.2) with arbitrary triangulations [11]:

1. The semi-discrete scheme (2.8), and certain time discretization of it, such as implicit backward Euler and Crank-Nicholson, have excellent nonlinear stability properties. One can prove a strong L^2 stability for the general nonlinear scalar case (1.2), for any orders of accuracy on arbitrary triangulations in any space dimension. This stability is independent of the size of the diffusion terms and hence is also valid in the limit when the diffusion coefficient goes to zero.
2. For linear problems with smooth solutions, these methods using piecewise polynomials of degree k have a provable error estimate of order Δx^k in L^2. In effect, for most triangulations one could observe (and prove in many cases) convergence of the order Δx^{k+1} in both L^2 and L^∞ norms, for both linear and nonlinear problems. The same error estimate can be obtained also for the auxiliary variable q, which approximates the derivative U_x, even if q can be locally eliminated in actual calculation.

2.3. KdV like equations. The one dimensional version of the KdV like equation (1.3) has the form

$$U_t + f(U)_x + (r'(U)g(r(U)_x)_x)_x = 0, \tag{2.10}$$

where $f(U)$, $r(U)$ and $g(U)$ are arbitrary functions.

The semi-discrete version of the local discontinuous Galerkin method for solving (2.10) [20] approximates the following lower order system

$$U_t + (f(U) + r'(U)P)_x = 0, \quad P - g(Q)_x = 0, \qquad Q - r(U)_x = 0. \tag{2.11}$$

We can then *formally* use the same discontinuous Galerkin method for the convection equation to solve (2.10), resulting in the following scheme: find $u, p, q \in \mathcal{V}_{\Delta x}$ such that, for all test functions $v, w, z \in \mathcal{V}_{\Delta x}$,

$$\begin{aligned}
&\int_{I_j} u_t v dx - \int_{I_j} (f(u) + r'(u)p)v_x dx + \left(\hat{f} + \widehat{r'}\hat{p}\right)_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - \left(\hat{f} + \widehat{r'}\hat{p}\right)_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0,\\
&\int_{I_j} pw dx + \int_{I_j} g(q)w_x dx - \hat{g}_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} + \hat{g}_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} = 0,\\
&\int_{I_j} qz dx + \int_{I_j} r(u)z_x dx - \hat{r}_{j+\frac{1}{2}} z^-_{j+\frac{1}{2}} + \hat{r}_{j-\frac{1}{2}} z^+_{j-\frac{1}{2}} = 0.
\end{aligned} \tag{2.12}$$

Again, a crucial ingredient for the method to be stable is the correct choice of the numerical fluxes (the "hats"). It is found out in [20] that one can take the following simple choice of fluxes to guarantee stability and convergence:

$$\hat{f} = \hat{f}(u^-, u^+), \quad \widehat{r'} = \frac{r(u^+) - r(u^-)}{u^+ - u^-}, \quad \hat{p} = p^+, \quad \hat{g} = \hat{g}(q^-, q^+), \quad \hat{r} = r(u^-) \tag{2.13}$$

where $\hat{f}(u^-, u^+)$ is a monotone flux for $f(u)$, and $-\hat{g}(q^-, q^+)$ is a monotone flux for $-g(q)$. In fact, the crucial part is still the "alternating principle" to take $\hat{p}$ and $\hat{r}$ from opposite sides. Thus

$$\hat{f} = \hat{f}(u^-, u^+), \quad \widehat{r'} = \frac{r(u^+) - r(u^-)}{u^+ - u^-}, \quad \hat{p} = p^-, \quad \hat{g} = \hat{g}(q^-, q^+), \quad \hat{r} = r(u^+)$$

would also work. Again, the appearance of the auxiliary variables p and q is superficial: when a local basis is chosen in cell I_j then both of them can be eliminated by using the second and third equations in (2.12) and solving two small linear systems if the local basis is not orthogonal. The actual scheme for u takes a form similar to that for convection alone. We also remark that the choice of fluxes in (2.13) by the alternating principle yields a compact stencil for the scheme for u after the auxiliary variables p and q are locally eliminated.

The schemes thus designed for the KdV like equation (2.10), or in fact for the most general multi dimensional nonlinear KdV like equations (1.3), which is nonlinear in all the derivatives, retain *all* of the three attractive properties listed above for the method used on convection equations. They also have the following provable theoretical properties, all of these are valid for the multi-dimensional case (1.3) with arbitrary triangulations [20]:

1. The semi-discrete scheme (2.12), and certain time discretization of it, such as implicit backward Euler and Crank-Nicholson, have excellent nonlinear stability properties. One can prove a strong L^2 stability and a cell entropy inequality for the square entropy, for the general nonlinear scalar case (1.3), for any orders of accuracy on arbitrary triangulations in any space dimension, without the need for nonlinear limiters [20]. Notice that these stability results are valid even in the limit when the coefficients of the dispersive third derivative terms tend to zero.
2. For one dimensional linear problems with smooth solutions, these methods using piecewise polynomials of degree k have a provable error estimate of order $\Delta x^{k+\frac{1}{2}}$ in L^2 [20]. In numerical experiments one observes convergence of the order Δx^{k+1} in both L^2 and L^∞ norms for both one and multiple dimensional linear and nonlinear cases.

3. A local discontinuous Galerkin method for the bi-harmonic type equations. In this section, we present and analyze a LDG method for the bi-harmonic type equation (1.4). We will concentrate on the one dimensional case

$$U_t + f(U)_x + (a(U_x)U_{xx})_{xx} = 0, \quad 0 \le x \le 1 \tag{3.1}$$

with an initial condition

$$U(x, 0) = U^0(x), \quad 0 \le x \le 1 \tag{3.2}$$

and periodic boundary conditions. Here $a(U_x) \ge 0$. The assumption of periodic boundary conditions is for simplicity only and is not essential: the method can be easily designed for non-periodic boundary conditions. The generalization to the multiple dimensional case (1.4) is straightforward, following the lines in [11] and [20].

To define the LDG method, we first introduce the new variables

$$R = U_x, \qquad Q = B(R)_x, \qquad P = (b(R)Q)_x, \tag{3.3}$$

where $b(R) = \sqrt{a(R)}$ and $B(R) = \int^R b(R)dR$, and rewrite the equation (3.1) as a first order system:

$$U_t + (f(U) + P)_x = 0, \quad P - (b(R)Q)_x = 0, \qquad Q - B(R)_x = 0, \quad R - U_x = 0. \tag{3.4}$$

The LDG method is obtained by discretizing the above system with the discontinuous Galerkin method. This is achieved by multiplying the four equations in (3.4) by four test functions v, w, z, s respectively, integrate over the interval I_j, and integrate by parts. We need to pay special attention to the boundary terms resulting from the procedure of integration by parts. Thus we seek piecewise polynomial solutions $u, p, q, r \in \mathcal{V}_{\Delta x}$, where $\mathcal{V}_{\Delta x}$ is defined in (2.3), such that for all test functions $v, w, z, s \in \mathcal{V}_{\Delta x}$ we have, for $1 \le j \le N$,

$$\begin{aligned}
\int_{I_j} u_t v dx - \int_{I_j} (f(u)+p) v_x dx + \left(\hat{f}+\hat{p}\right)_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - \left(\hat{f}+\hat{p}\right)_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0, \\
\int_{I_j} p w dx + \int_{I_j} b(r) q w_x dx - \left(\hat{b}\hat{q}\right)_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} + \left(\hat{b}\hat{q}\right)_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} = 0, \\
\int_{I_j} q z dx + \int_{I_j} B(r) z_x dx - \hat{B}_{j+\frac{1}{2}} z^-_{j+\frac{1}{2}} + \hat{B}_{j-\frac{1}{2}} z^+_{j-\frac{1}{2}} = 0, \\
\int_{I_j} r s dx + \int_{I_j} u s_x dx - \hat{u}_{j+\frac{1}{2}} s^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}} s^+_{j-\frac{1}{2}} = 0.
\end{aligned} \tag{3.5}$$

The only ambiguity in the algorithm (3.5) now is the definition of the numerical fluxes (the "hats"), which should be designed to ensure stability. It turns out that we can take the simple choices (we omit the subscripts $j \pm \frac{1}{2}$ in the definition of the fluxes as all quantities are evaluated at the interfaces $x_{j\pm\frac{1}{2}}$)

$$\hat{f} = \hat{f}(u^-, u^+), \quad \hat{p} = p^-, \quad \hat{b} = \frac{B(r^+) - B(r^-)}{r^+ - r^-}, \quad \hat{q} = q^+, \qquad \hat{B} = B(r^-), \quad \hat{u} = u^+ \tag{3.6}$$

where $\hat{f}(u^-, u^+)$ is a monotone flux for $f(u)$ in (2.4). Here again, the "alternating principle" in designing the fluxes related to the fourth derivatives is at play: the fluxes for p, q, $B(r)$ and u are alternatively taken from left and right. We could thus also take

$$\hat{f} = \hat{f}(u^-, u^+), \quad \hat{p} = p^+, \quad \hat{b} = \frac{B(r^+) - B(r^-)}{r^+ - r^-}, \quad \hat{q} = q^-, \qquad \hat{B} = B(r^+), \quad \hat{u} = u^- \tag{3.7}$$

and stability is still valid.

We remark again that the appearance of the auxiliary variables p, q and r is superficial: when a local basis is chosen in cell I_j then all of them can be eliminated by using the second, third, and fourth equations in (3.5) and solving three small linear systems if the local basis is not orthogonal. The actual scheme for u takes a form similar to that for convection alone.

We have the following L^2 stability result for the scheme (3.5)-(3.6). This is similar to the stability result in Proposition 2.1 in [11] for the second order convection diffusion equations.

Proposition 3.1. (L^2 stability) The solution of the scheme (3.5)-(3.6) satisfies

$$\frac{d}{dt} \int_0^1 \left(\frac{u^2(x,t)}{2} \right) dx + \int_0^1 q^2(x,t) dx \le 0. \tag{3.8}$$

Proof: We sum up the four equalities in (3.5) and introduce the notation

$$\begin{aligned}
B_j(u,p,q,r;v,w,z,s) = & \int_{I_j} u_t v dx - \int_{I_j} (f(u)+p) v_x dx + \left(\hat{f}+\hat{p}\right)_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} \\
& - \left(\hat{f}+\hat{p}\right)_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} + \int_{I_j} p w dx + \int_{I_j} b(r) q w_x dx - \left(\hat{b}\hat{q}\right)_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} \\
& + \left(\hat{b}\hat{q}\right)_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} + \int_{I_j} q z dx + \int_{I_j} B(r) z_x dx - \hat{B}_{j+\frac{1}{2}} z^-_{j+\frac{1}{2}} \\
& + \hat{B}_{j-\frac{1}{2}} z^+_{j-\frac{1}{2}} + \int_{I_j} r s dx + \int_{I_j} u s_x dx - \hat{u}_{j+\frac{1}{2}} s^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}} s^+_{j-\frac{1}{2}}.
\end{aligned} \tag{3.9}$$

Clearly, the solutions u, p, q, r of the scheme (3.5)-(3.6) satisfy

$$B_j(u,p,q,r;v,w,z,s)=0 \tag{3.10}$$

for all $v,w,z,s\in\mathcal{V}_{\Delta x}$. We then take

$$v=u,\qquad w=r,\qquad z=q,\qquad s=-p$$

to obtain, after some algebraic manipulations,

$$\begin{aligned}0&=B_j(u,p,q,r;u,r,q,-p)\\&=\frac{d}{dt}\int_{I_j}\left(\frac{u^2(x,t)}{2}\right)dx+\int_{I_j}q^2(x,t)dx+\left(\hat{H}_{j+\frac{1}{2}}-\hat{H}_{j-\frac{1}{2}}\right)+\Theta_{j-\frac{1}{2}}\end{aligned} \tag{3.11}$$

with the numerical entropy flux $\hat{H}$ defined by

$$\hat{H}=-F(u^-)-p^-u^-+B(r^-)q^-+\left(\hat{f}+\hat{p}\right)u^--\left(\hat{b}\hat{q}\right)r^--\hat{B}q^-+\hat{u}p^-$$

and the extra term Θ given by

$$\Theta=[F(u)+pu-B(r)q]-\left(\hat{f}+\hat{p}\right)[u]+\hat{b}\hat{q}[r]+\hat{B}[q]-\hat{u}[p],$$

Here $F(u)=\int^u f(u)du$, and $[v]=v^+-v^-$ denotes the jump of v. Notice that we have dropped the subscripts about the location $j-\frac{1}{2}$ or $j+\frac{1}{2}$ as all these quantities are defined at a single interface and depend only on the left and right values at that interface. Now all we need to do is to verify $\Theta\geq 0$. To this end, we notice that, with the definition (3.6) of the numerical fluxes and with simple algebraic manipulations, we easily obtain

$$[pu-B(r)q]-\hat{p}[u]+\hat{b}\hat{q}[r]+\hat{B}[q]-\hat{u}[p]=0$$

and hence

$$\Theta=[F(u)]-\hat{f}[u]=\int_{u^-}^{u^+}\left(f(s)-\hat{f}(u^-,u^+)\right)ds\geq 0,$$

because of the monotonicity of the flux $\hat{f}$. Summing up (3.11) over j would now give the desired L^2 stability (3.8). ■

For a preliminary numerical example of the methods developed in this section, see Example 5.3 in section 5.

4. A local discontinuous Galerkin method for equations with fifth derivatives. In this section, we present and analyze a LDG method for the equation (1.5) with fifth derivatives. We will concentrate on the one dimensional case

$$U_t+f(U)_x+g(U_{xx})_{xxx}\ =\ 0,\quad 0\leq x\leq 1 \tag{4.1}$$

with an initial condition

$$U(x,0)=U^0(x),\quad 0\leq x\leq 1 \tag{4.2}$$

and periodic boundary conditions. Here $g(U_{xx})$ is an arbitrary function of U_{xx}. The form of the nonlinearity is not general and is chosen simply as an example. The assumption of periodic boundary conditions is also for

simplicity only and is not essential: the method can be easily designed for non-periodic boundary conditions. The generalization to the multiple dimensional case (1.5) is straightforward, following the lines in [11] and [20].

To define the LDG method, we first introduce the new variables

$$S = U_x, \qquad R = S_x, \qquad Q = g(R)_x, \qquad P = Q_x, \tag{4.3}$$

and rewrite the equation (4.1) as a first order system:

$$U_t + (f(U) + P)_x = 0, \quad P - Q_x = 0, \qquad Q - g(R)_x = 0, \quad R - S_x = 0, \quad S - U_x = 0. \tag{4.4}$$

The LDG method is obtained by discretizing the above system with the discontinuous Galerkin method. This is achieved by multiplying the five equations in (4.4) by five test functions v, w, y, z, d respectively, integrate over the interval I_j, and integrate by parts. We need to pay special attention to the boundary terms resulting from the procedure of integration by parts. Thus we seek piecewise polynomial solutions $u, p, q, r, s \in \mathcal{V}_{\Delta x}$, where $\mathcal{V}_{\Delta x}$ is defined in (2.3), such that for all test functions $v, w, y, z, d \in \mathcal{V}_{\Delta x}$ we have, for $1 \le j \le N$,

$$\begin{aligned}
\int_{I_j} u_t v dx - \int_{I_j} (f(u) + p) v_x dx + \left(\hat{f} + \hat{p}\right)_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - \left(\hat{f} + \hat{p}\right)_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0, \\
\int_{I_j} p w dx + \int_{I_j} q w_x dx - \hat{q}_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} + \hat{q}_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} = 0, \\
\int_{I_j} q y dx + \int_{I_j} g(r) y_x dx - \hat{g}_{j+\frac{1}{2}} y^-_{j+\frac{1}{2}} + \hat{g}_{j-\frac{1}{2}} y^+_{j-\frac{1}{2}} = 0, \\
\int_{I_j} r z dx + \int_{I_j} s z_x dx - \hat{s}_{j+\frac{1}{2}} z^-_{j+\frac{1}{2}} + \hat{s}_{j-\frac{1}{2}} z^+_{j-\frac{1}{2}} = 0, \\
\int_{I_j} s d\, dx + \int_{I_j} u d_x\, dx - \hat{u}_{j+\frac{1}{2}} d^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}} d^+_{j-\frac{1}{2}} = 0.
\end{aligned} \tag{4.5}$$

The only ambiguity in the algorithm (4.5) now is the definition of the numerical fluxes (the "hats"), which should be designed to ensure stability. It turns out that we can again take the simple choices

$$\hat{f} = \hat{f}(u^-, u^+), \quad \hat{p} = p^-, \quad \hat{q} = q^+, \quad \hat{g} = \hat{g}(r^-, r^+), \qquad \hat{s} = s^-, \quad \hat{u} = u^+ \tag{4.6}$$

where $\hat{f}(u^-, u^+)$ is a monotone flux for $f(u)$ in (2.4), and $-\hat{g}(r^-, r^+)$ is a monotone flux for $-g(r)$. Here again, the "alternating principle" in designing the fluxes related to the fifth derivatives is at play: the fluxes for p, q, s and u are alternatively taken from left and right. We could thus also take

$$\hat{f} = \hat{f}(u^-, u^+), \quad \hat{p} = p^+, \quad \hat{q} = q^-, \quad \hat{g} = \hat{g}(r^-, r^+), \qquad \hat{s} = s^+, \quad \hat{u} = u^- \tag{4.7}$$

and stability is still valid.

As before, the appearance of the auxiliary variables p, q, r and s is superficial: when a local basis is chosen in cell I_j then all of them can be eliminated by using the second, third, fourth and fifth equations in (4.5) and solving four small linear systems if the local basis is not orthogonal. The actual scheme for u takes a form similar to that for convection alone.

We have the following cell entropy inequality for the square entropy, which implies L^2 stability, when summed up over j, for the scheme (4.5)-(4.6). This is similar to the cell entropy inequality in [14] for conservation laws and the one in [20] for KdV type equations involving third derivatives.

Proposition 4.1. (cell entropy inequality) The solution of the scheme (4.5)-(4.6) satisfies

$$\frac{d}{dt}\int_{I_j}\left(\frac{u^2(x,t)}{2}\right)dx+\left(\hat{H}_{j+\frac{1}{2}}-\hat{H}_{j-\frac{1}{2}}\right)\leq 0 \tag{4.8}$$

for some numerical entropy flux $\hat{H}_{j+\frac{1}{2}}$.

Proof: We sum up the five equalities in (4.5) and introduce the notation

$$\begin{aligned}
B_j(u,p,q,r,s;v,w,y,z,d) = &\int_{I_j} u_t v dx - \int_{I_j}(f(u)+p)v_x dx + \left(\hat{f}+\hat{p}\right)_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} \\
&- \left(\hat{f}+\hat{p}\right)_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} + \int_{I_j} pw dx + \int_{I_j} q w_x dx - \hat{q}_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} \\
&+\hat{q}_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} + \int_{I_j} qy dx + \int_{I_j} g(r) y_x dx - \hat{g}_{j+\frac{1}{2}} y^-_{j+\frac{1}{2}} \\
&+\hat{g}_{j-\frac{1}{2}} y^+_{j-\frac{1}{2}} + \int_{I_j} rz dx + \int_{I_j} s z_x dx - \hat{s}_{j+\frac{1}{2}} z^-_{j+\frac{1}{2}} + \hat{s}_{j-\frac{1}{2}} z^+_{j-\frac{1}{2}} \\
&+ \int_{I_j} s d\, dx + \int_{I_j} u d_x\, dx - \hat{u}_{j+\frac{1}{2}} d^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}} d^+_{j-\frac{1}{2}}.
\end{aligned} \tag{4.9}$$

Clearly, the solutions u, p, q, r and s of the scheme (4.5)-(4.6) satisfy

$$B_j(u,p,q,r,s;v,w,y,z,d)=0 \tag{4.10}$$

for all $v,w,y,z,d\in\mathcal{V}_{\Delta x}$. We then take

$$v=u, \qquad w=s, \qquad y=-r, \qquad z=q, \qquad d=-p$$

to obtain, after some algebraic manipulations,

$$0=B_j(u,p,q,r,s;u,s,-r,q,-p)=\frac{d}{dt}\int_{I_j}\left(\frac{u^2(x,t)}{2}\right)dx+\left(\hat{H}_{j+\frac{1}{2}}-\hat{H}_{j-\frac{1}{2}}\right)+\Theta_{j-\frac{1}{2}} \tag{4.11}$$

with the numerical entropy flux $\hat{H}$ defined by

$$\hat{H}=-F(u^-)-p^-u^-+q^-s^--G(r^-)+\left(\hat{f}+\hat{p}\right)u^--\hat{q}s^-+\hat{g}r^--\hat{s}q^-+\hat{u}p^-$$

and the extra term Θ given by

$$\Theta=[F(u)+pu-qs+G(r)]-\left(\hat{f}+\hat{p}\right)[u]+\hat{q}[s]-\hat{g}[r]+\hat{s}[q]-\hat{u}[p],$$

Here $F(u)=\int^u f(u)du$ and $G(r)=\int^r g(r)dr$, and $[v]=v^+-v^-$ denotes the jump of v. Now all we need to do is to verify $\Theta\geq 0$. To this end, we notice that, with the definition (4.6) of the numerical fluxes and with simple algebraic manipulations, we easily obtain

$$[pu-qs]-\hat{p}[u]+\hat{q}[s]+\hat{s}[q]-\hat{u}[p]=0$$

and hence

$$\begin{aligned}
\Theta &= [F(u)]-\hat{f}[u]-[G(r)]+\hat{g}[r] \\
&= \int_{u^-}^{u^+}\left(f(\xi)-\hat{f}(u^-,u^+)\right)d\xi-\int_{r^-}^{r^+}\left(g(\xi)-\hat{g}(r^-,r^+)\right)d\xi \\
&\geq 0,
\end{aligned}$$

where the last inequality follows from the monotonicity of the fluxes $\hat{f}$ and $-\hat{g}$. This finishes the proof. ■

For a preliminary numerical example of the methods developed in this section, see Example 5.4 in section 5.

5. Post-processing and numerical results. In [7], an efficient local post-processing technique has been developed, which is applied only at the end of the calculation to the numerical solution at $t = T$, and involves only a local linear operation (convolution) using the information from a fixed (independent of the mesh size Δx) number of neighboring cells. The description of this post-processing procedure is omitted here because of space limitation. Please see [7] for details.

It is proven in [7] that this post processing will improve the order of accuracy from Δx^{k+1} to Δx^{2k+1} when piecewise polynomials of degree k are used, for linear conservation laws (1.1) with a uniform (or locally uniform) mesh with smooth solutions. Although not proven theoretically, it is also verified numerically in [7] that the same enhancement of accuracy can be observed for linear convection diffusion equations (1.2). In fact, the crucial ingredient this post-processing technique depends on is a negative-order error estimate, which is usually higher than the order of convergence of the L^2-norm of the error, for Galerkin type methods. If the problem is linear, the solution is sufficiently smooth, then the post-processing will achieve the order of convergence of the negative-order norm.

In this section, we perform numerical experiments of this post-processing techniques when the local discontinuous Galerkin method is applied to the linear and nonlinear KdV equations (1.3), and linear versions of (1.4) and (1.5) containing fourth and fifth derivatives. We observe the same accuracy enhancement. This strongly suggests that the LDG methods we have designed for these equations with higher derivatives have an order of convergence of $2k+1$ or higher in the negative-order norm, when piecewise polynomials of degree k are used. The post-processing technique thus is fully taking advantage of this. Time discretization is by the third and fourth order implicit Runge-Kutta methods in [13]. An efficient time marching of these local discontinuous Galerkin methods without sacrificing its local property and parallel flexibility is left for future work. We have chosen Δt suitably small so that spatial errors dominate in the numerical results.

Example 5.1. We solve the linear KdV equation (1.3) in 1D:

$$U_t + U_{xxx} = 0 \tag{5.1}$$

with initial condition $U(x,0) = \sin(x)$ and periodic boundary conditions. The exact solution is given by $U(x,t) = \sin(x+t)$. The numerical errors and order of accuracy before and after post-processing can be found in Table 5.1. We clearly observe an accuracy of Δx^{k+1} before the post-processing and Δx^{2k+1} or higher (closer to Δx^{2k+2}) after it, when piecewise P^k elements are used. We do not present the result with $k = 0$ because for this case the results before the post-processing is already shown in [20] and the post-processing does not increase the order of accuracy ($2k+1 = k+1$ for $k = 0$).

Example 5.2. We compute the classical soliton solution of the nonlinear KdV equation (1.3) in 1D:

$$U_t - 3\left(U^2\right)_x + U_{xxx} = 0 \tag{5.2}$$

with an initial condition $u(x,0) = -2\,\mathrm{sech}^2(x)$ for $-14 \le x \le 16$, see [20] for details. The exact solution is given by $U(x,t) = -2\,\mathrm{sech}^2(x-4t)$. The numerical errors and order of accuracy before and after post-processing can be found in Table 5.2. We clearly observe an accuracy of Δx^{k+1} before the post-processing and Δx^{2k+1} or higher after it, when piecewise P^k elements are used. This indicates that the post-processing technique is effective in enhancing orders of accuracy even for this nonlinear PDE.

Example 5.3. We solve the linear bi-harmonic equation (1.4) in 1D:

$$U_t + U_{xxxx} = 0 \tag{5.3}$$

TABLE 5.1

$U_t + U_{xxx} = 0$. $U(x,0) = \sin(x)$. Periodic boundary conditions. L^∞ errors. Uniform meshes with N cells. LDG methods with $k = 1, 2, 3$. $t = 1$. Before (B) and after (A) post-processing.

k		**N=10**	**N=20**		**N=40**		**N=80**	
		error	error	order	error	order	error	order
1	*B*	5.0265E-02	1.3623E-02	1.88	1.0619E-03	1.96	8.8570E-04	1.98
	A	5.7711E-03	5.8787E-04	3.29	6.3866E-05	3.20	7.3564E-06	3.11
2	*B*	2.9084E-03	3.6532E-04	2.99	4.6186E-05	2.98	5.7816E-06	2.99
	A	2.0588E-04	3.6032E-06	5.83	6.3684E-08	5.82	1.2763E-09	5.64
		N=10	**N=20**		**N=40**		**N=50**	
3	*B*	9.2247E-05	6.0315E-06	3.93	3.8021E-07	3.98	1.5583E-07	3.99
	A	2.2816E-05	9.7519E-08	7.87	3.9135E-10	7.96	7.0005E-11	7.71

TABLE 5.2

$U_t - 3\left(U^2\right)_x + U_{xxx} = 0$. $U(x,0) = -2\,sech^2(x)$. L^∞ errors. Uniform meshes with N cells. LDG methods with $k = 1, 2, 3$. $t = 0.5$. Before (B) and after (A) post-processing.

k		**N=80**	**N=160**		**N=320**		**N=640**	
		error	error	order	error	order	error	order
1	*B*	7.1035E-02	1.4334E-02	2.30	4.3454E-03	1.72	1.1852E-03	1.87
	A	4.8823E-02	6.0560E-03	3.01	7.3566E-04	3.04	8.9456E-05	3.03
2	*B*	4.8540E-03	6.2916E-04	2.94	8.0172E-05	2.97	1.0047E-05	2.99
	A	3.0621E-03	8.0491E-05	5.24	1.6184E-06	5.63	3.3110E-08	5.61
		N=80	**N=160**		**N=320**		**N=500**	
3	*B*	3.5385E-04	2.4394E-05	3.85	1.5343E-06	3.99	2.5907E-07	3.98
	A	1.6221E-03	1.6713E-05	6.60	8.9620E-08	7.54	3.3679E-09	7.35

with initial condition $U(x,0) = \sin(x)$ and periodic boundary conditions. The exact solution is given by $U(x,t) = e^{-t}\sin(x)$. The numerical errors and order of accuracy before and after post-processing can be found in Table 5.3. We clearly observe an accuracy of Δx^{k+1} before the post-processing and Δx^{2k+1} or higher (closer to Δx^{2k+2}) after it, when piecewise P^k elements are used. We have used multiple precision FORTRAN to compute part of this problem, with an effective precision higher than the standard double precision, to overcome some of the round-off problems for this problem. If such multiple precision procedure is not used, then the post-processed error can only go down to around 10^{-10} level and cannot decrease further. The result in Table 5.3 verifies both the stability and accuracy of the local discontinuous Galerkin method developed in section 3 and also the effectiveness of the post-processing technique for such methods applied to this type of equations. It also strongly suggests that there is a better negative norm error estimate on the order of Δx^{2k+2} for piecewise P^k elements to this problem.

Example 5.4. We solve the linear equation (1.5) in 1D:

$$U_t + U_{xxxxx} = 0 \tag{5.4}$$

with initial condition $U(x,0) = \sin(x)$ and periodic boundary conditions. The exact solution is given by

TABLE 5.3

$U_t + U_{xxxx} = 0$. $U(x,0) = \sin(x)$. Periodic boundary conditions. L^∞ errors. Uniform meshes with N cells. LDG methods with $k = 0, 1, 2, 3$. $t = 1$. Before (B) and after (A) post-processing.

k		**N=10**	**N=20**		**N=40**		**N=80**	
		error	error	order	error	order	error	order
0	*B*	1.1125E-01	5.4352E-02	1.03	2.7001E-02	1.00	1.3478E-02	1.00
1	*B*	2.2038E-02	5.2262E-03	2.07	1.3119E-03	1.99	3.2831E-04	1.99
	A	7.5566E-04	4.5478E-05	4.05	2.8624E-06	3.98	1.7930E-07	3.99
2	*B*	1.1183E-03	1.3512E-04	3.04	1.6988E-05	2.99	2.1265E-06	2.99
	A	9.4333E-05	1.2978E-06	6.18	1.9072E-08	6.08	2.9261E-10	6.02
3	*B*	6.1004E-05	2.3484E-06	4.69	1.4022E-07	4.06	8.7476E-09	4.00
	A	3.2399E-05	1.6632E-07	7.60	6.9985E-10	7.89	2.7864E-12	7.97

TABLE 5.4

$U_t + U_{xxxxx} = 0$. $U(x,0) = \sin(x)$. Periodic boundary conditions. L^∞ errors. Uniform meshes with N cells. LDG methods with $k = 0, 1, 2, 3$. $t = 1$. Before (B) and after (A) post-processing.

k		**N=10**	**N=20**		**N=40**		**N=80**	
		error	error	order	error	order	error	order
0	*B*	5.6284E-01	2.5073E-01	1.16	1.1647E-01	1.10	5.5926E-02	1.05
1	*B*	4.8495E-02	1.3661E-02	1.82	3.5772E-03	1.93	8.9887E-04	1.99
	A	6.6528E-03	6.3604E-04	3.38	6.2508E-05	3.34	6.7327E-06	3.21
2	*B*	2.9071E-03	3.6668E-04	2.98	4.6236E-05	2.98	5.7831E-06	2.99
	A	2.3284E-04	2.8382E-06	6.35	3.7362E-08	6.24	3.7811E-10	6.62
3	*B*	1.4253E-04	6.0409E-06	4.56	3.8018E-07	3.99	2.3783E-08	3.99
	A	1.0433E-04	4.1471E-07	7.97	1.6121E-09	8.00	6.2839E-12	8.00

$U(x,t) = \sin(x-t)$. The numerical errors and order of accuracy before and after post-processing can be found in Table 5.4. We clearly observe an accuracy of Δx^{k+1} before the post-processing and Δx^{2k+2} (one order higher than the expected Δx^{2k+1}) after it, when piecewise P^k elements are used. This strongly suggests that there is a better negative norm error estimate on the order of Δx^{2k+2} (at least for $k = 2$ and 3 among the tested cases) for piecewise P^k elements to this problem. We have used multiple precision FORTRAN to compute part of this problem, with an effective precision higher than the standard double precision, to overcome some of the round-off problems for this problem. If such multiple precision procedure is not used, then the post-processed error can only go down to around 10^{-8} to 10^{-9} level and cannot decrease further. The result in Table 5.4 clearly verifies both the stability and accuracy of the local discontinuous Galerkin method developed in section 4 and also the effectiveness of the post-processing technique for such methods applied to this type of equations.

6. Concluding remarks. We have reviewed local discontinuous Galerkin methods for solving PDEs with first, second and third spatial derivatives and developed new local discontinuous Galerkin methods for solving PDEs with fourth derivatives (the time dependent bi-harmonic equations) and those with fifth derivatives. We have designed the numerical fluxes and proved that such methods are L^2 stable and satisfy

cell entropy inequalities for general nonlinear PDEs. Although the discussion is concentrate in 1D because of space limitation, the results can be easily generalized to multiple spatial dimensions. Error estimates for linear equations similar to those offered in [11] and [20] can be obtained along similar lines but are not given in this paper. PDEs with even higher spatial derivatives can also be handled along similar lines. We also apply a post processing technique and demonstrate that the accuracy can be improved from Δx^{k+1} to Δx^{2k+1} or higher when piecewise polynomials of degree k are used. In future work we will investigate efficient time marching of these local discontinuous Galerkin methods without sacrificing its local property and parallel flexibility, and perform more numerical experiments with physically interesting problems.

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In this paper we review the existing and develop new local discontinuous Galerkin methods for solving time dependent partial differential equations with higher order derivatives in one and multiple space dimensions. We review local discontinuous Galerkin methods for convection diffusion equations involving second derivatives and for KdV type equations involving third derivatives. We then develop new local discontinuous Galerkin methods for the time dependent bi-harmonic type equations involving fourth derivatives, and partial differential equations involving fifth derivatives. For these new methods we present correct interface numerical fluxes and prove L^2 stability for general nonlinear problems. Preliminary numerical examples are shown to illustrate these methods. Finally, we present new results on a post-processing technique, originally designed for methods with good negative-order error estimates, on the local discontinuous Galerkin methods applied to equations with higher derivatives. Numerical experiments show that this technique works as well for the new higher derivative cases, in effectively doubling the rate of convergence with negligible additional computational cost, for linear as well as some nonlinear problems, with a local uniform mesh.

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